measurement, theory of. Most mathematical sciences use quantitative models, and the theory of measurement is devoted to making explicit the qualitative assumptions that give rise to them. This is accomplished by first stating the qualitative assumptions – empirical laws of the most elementary sort – in axiomatic form and then showing that there are structure preserving mappings, often but not always isomorphisms, from the qualitative structure into the quantitative one. The set of such mappings forms what is called a *scale* of measurement.

A theory of the possible numerical scales plays an important role throughout measurement – and therefore throughout science – since, just as the qualitative assumptions of a class of structures narrowly determine the nature of the possible scales, so also the nature of the underlying scales greatly limits the possible qualitative structures that give rise to such scales. Our two major themes, which reflect relatively new research results, are, first, that the possible scales that are useful in science are necessarily very limited and, second that once a type of scale is selected (or assumed to exist) for a qualitative structure, then a great deal is known about that structure and its quantitative models.

There are several general references to the axiomatic theory. Perhaps the most elementary and the one with the most examples is Roberts (1979). Pfanzagl (1968) and Krantz et al. (1971) are on a par, with the latter more comprehensive. Narens (1985) is the mathematically most sophisticated, and covers much of the material mentioned here. We cite only references not included in one of these surveys.

AXIOMATIZABILITY

The Qualitative Setup. The qualitative situation is usually conceptualized as a *relational structure* $\mathscr{X} = \langle X, S_0, S_1, \ldots \rangle$, where the $S_0, S_1, \ldots$ are relations of finite order on X. The set of relations can be either finite or infinite. X is called the *domain* of the structure and the S_i its *primitive relations*. In most applications, S_0 will be some type of ordering relation, and when this is the case it will be written as $\succsim$. The following are some examples of qualitative structures used in measurement situations. The first, which goes back to Helmholtz, has for its domain a set X of objects with the property of having mass. There are two primitive relations. The first, $\succsim$, is a binary ordering according to mass (which may be determined, for example, by using an equal-arm pan balance so that $x \succsim y$ means that the pans either remain level or the one containing x drops). The second relation, $\circ$, is a ternary one that can be interpreted as a binary operation. Empirically, it is defined as follows: if x and y are placed in the same pan and are exactly balanced by z, then we write $x \circ y \sim z$, where $\sim$ means equivalence in the attribute. The structure $\langle X, \succsim, \circ \rangle$ was used by Helmholtz in developing an axiomatic treatment of the measurement of mass.

A second example is from economics. Suppose $C_1, \ldots, C_n$ are sets each consisting of different amounts of a commodity, and $\succsim$ is a preference ordering exhibited by a person or an institution over the set of possible commodity bundles $C = X_i C_i$. The resulting structure $\mathscr{E} = \langle C, \succsim \rangle$, known as a *conjoint* one, can among other things be used to induce ordering of an individual's preferences for the commodities associated with each component.

The third example, due to B. de Finetti, has as its domain an algebra $\mathscr{E}$ of subsets, called 'events', of some non-empty set Ω. The primitives of the structure consist of an ordering relation $\succsim$ of 'at least as likely as', the events Ω and $\varnothing$, and the set theoretical operations of union $\cup$, intersection $\cap$, and complementation $\sim$. The relational structure

$$\mathscr{P} = \langle \mathscr{E}, \succsim, \Omega, \varnothing, \cup, \cap, \sim \rangle$$

is intended to characterize qualitatively probability-like situations. The primitive $\succsim$ can arise from many different processes, depending upon the situation. In one, which is of considerable importance to Bayesians, $\succsim$ represents a person's ordering of events according to how likely they seem using whatever basis he or she wishes in making the judgements. In such a case, $\mathscr{P}$; is thought of as a subjective or personal probability structure. In another $\succsim$ is based on some probability model for the situation (possibly one coupled with estimated relative frequencies), as in much of classical probability theory.

Representations and Scales. A key notion in the theory of measurement is that of a *representation*, which is defined to be a structure preserving map ϕ of the qualitative relational structure $\mathscr{X}$ into a quantitative one, $\mathscr{R}$, in which the domain is a subset of the real numbers. Representations are either isomorphisms or in cases where equivalences play an important role (e.g. conjoint structures where trade-off between components is the essence of the matter) as homomorphisms, in which case equivalence classes of equivalent elements are assigned the same number. We say 'ϕ is a $\mathscr{R}$-representation for $\mathscr{X}$'.

For the past three decades, measurement theorists have been exploring certain types of qualitative structures for which numerical representations exist. The questions faced are, first, to establish that the set of $\mathscr{R}$-representations is non-empty for The primitives of the structure consist of an ordering relation $\succsim$ of 'at least as likely as', the events Ω and $\varnothing$, and the set generate all of them once one is specified. The first is called the 'existence' problem and the second, the 'uniqueness' problem. Several examples will be cited.

For the qualitative mass structure $\mathscr{X} = \langle X, \succsim, \circ \rangle$ described above, the qualitative representing structure is taken to be $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, + \rangle$, where Re^+ is the positive real numbers, and $\geqslant$ and $+$ have their usual meanings in the real number system. The set of $\mathscr{R}$-representations of $\mathscr{X}$ consist of all functions ϕ from X into Re^+ such that for each x and y in X, (i) $x \succsim y$ iff $\phi(x) \geqslant \phi(y)$, and (ii) $\phi(x \circ y) = \phi(x) + \phi(y)$. Such a function is called a *homomorphism*, and the set of all of them is called a *scale*. In addition to Helmholtz, others including O. Hölder, P. Suppes, Luce and A. A. J. Marley and J.-C. Falmagne have stated axioms about the primitives of $\mathscr{X}$ that are sufficient to show the existence of such homomorphisms and to show that any two homomorphisms ϕ and ψ are related by multiplication, that is, there is some real $r > 0$ such that $\psi = r\phi$. In the language introduced later by S. S. Stevens (1946), such a form of measurement is said to be a 'ratio scale'. For the case where $\circ$ is an operation (defined for all pairs), F. S. Roberts (1979) and Luce and Narens (1985) have given necessary and sufficient conditions for such a representation. A complete characterization, such as this one, is rather unusual in measurement; sufficient conditions are far more the rule.

Representations of the structure $\mathscr{C} = \langle X_i C_i, \succsim \rangle$ of commodity bundles are usually taken in economics to be n-tuples $\langle \phi_1, \ldots, \phi_n \rangle$ of functions, where ϕ_i maps C_i into Re, such that for each x_i, y_i in C_i, $i = 1, \ldots, n$,

$$(x_1, \ldots, x_n) \succsim (y_1, \ldots, y_n) \quad \text{iff} \quad \sum_i \phi(x_i) \geqslant \sum_i \phi(y_i).$$

In the measurement literature such a representation is called 'additive'. G. Debreu, Luce and J. W. Tukey, D. Scott, A. Tversky and others, have given axioms on $\mathscr{C}$ for which existence of an additive representation can be shown, and that any two representations $(\phi_1, \ldots, \phi_n)$ and $(\psi_1, \ldots, \psi_n)$ are related by affine transformations of the form $\psi_i = r\phi_i + s_i$, $i = 1, \ldots, n$, $r > 0$. In Stevens's nomenclature, the set of such representations are said to form an 'interval scale'.

In the example of the subjective probability structure $\mathscr{P} = \langle \mathscr{E}, \succsim, \Omega, \varnothing, \cup, \cap, \sim \rangle$, the usual sort of representation is a probability function P from $\mathscr{E}$ into [0, 1], that is, for all A, B in $\mathscr{E}$, (i) $P(\Omega) = 1$, $P(\varnothing) = 0$, (ii) $A \succsim B$ iff $P(A) \geqslant P(B)$, (iii) if $A \cap B = \varnothing$, then $P(A \cup B) = P(A) + P(B)$.

A number of authors have given sufficient conditions in terms of the primitives for P to exist. Fine (1973) gives a good summary of a variety of approaches to probability, In the probability case, unlike the previous two, if P and Q are two representations for $\mathscr{P}$, then $P = Q$; that is, the representation is uniquely determined. Such scales were called 'absolute' by Stevens.

Empirical Usefulness of Axiomatic Treatments. One advantage of a measurement approach to some scientific questions is that it offers an alternative way of testing quantitative models other than simple goodness of fit. Since the axiomatic approach isolates a series of properties that are in some sense thought to be basic, it leads to the validation or invalidation of specific axioms rather than the entire model. In particular, this approach often makes clear where the source of the problem is and thus gives insight into how the model must be altered. An example of this, familiar to economists, has arisen in the theory of subjective expected utility. In its simplest form the domain is gambles of the form $x \circ_A y$ in which x is the outcome if event A occurs and y if A fails to occur, where x and y may themselves be gambles, and the theory postulates a preference ordering $\succsim$ over the outcomes and gambles. The classical axiomatization (for a summary, see Fishburn, 1970) establishes conditions on preferences over gambles so that there exists a probability measure P on the algebra of events, as in a probability structure, and a 'utility function' U over the gambles such that U preserves $\succsim$ and

$$U(x \circ_A y) = P(A)U(x) + [1 - P(A)]U(y). \qquad (1)$$

A series of empirical studies (for summaries see Allais and Hagen, 1979, and Kahneman and Tversky, 1979) have made clear that this representation, which can be readily defended on grounds of rationality, is inadequate to describe human behaviour. Among its axioms, the one that appears to be the major source of difficulty is the 'extended sure-thing principle'. It may be stated as follows: suppose A, B and C are events, with C disjoint from A and B, then

$$x \circ_A y \succsim x \circ_B y \quad \text{iff} \quad x \circ_{A \cup C} y \succsim x \circ_{B \cup C} y. \qquad (2)$$

It is easy to verify that this is a necessary condition if equation (1) holds, and it seems to be one that people are unwilling to abide by. Any modification of the theory that is to be descriptive of human behaviour must abandon it.

A related example of the interplay of axioms and data, also of interest to economists, is the measurement of risk (references can be found in Weber, 1984; Luce and Weber, 1986).

It must be acknowledged that the isolation of properties in the axiomatic approach has an apparently happenstance quality. The choice of axioms for an empirical structure is by no means uniquely determined; there is an infinity of equivalent axiom systems for any infinite structure, and it is by no means clear why we tend to select the ones we do. It is entirely possible for the failure of the model to be described easily in terms of one axiomatization, and to be totally obscure in another. Furthermore, not everyone values the overall axiomatic approach to scientific (in contrast to mathematical) questions; in particular, it has been sharply attacked by Anderson (1981, pp. 347–56).

Another use of axiomatic methods and of the notion of scale (see *Representations and Scales* above) is in the study of meaningfulness, which is treated under MEANINGFULNESS AND INVARIANCE.

2. ORDERED STRUCTURES

Two types of 'quantitative' representations have played a major role in science: systems of coordinate geometry and the real number system (the latter being the one-dimensional specialization of the former). The latter is our focus. The absolutely simplest case, included in all of the above examples, is the representation of $\langle X, \succsim \rangle$ into $\langle \text{Re}, \geqslant \rangle$, that is, the case where there is a mapping ϕ from X into Re such that $x \succsim y$ iff $\phi(x) \succsim \phi(y)$. It follows readily that in such situations $\succsim$ must be transitive, connected, and reflexive. Such relations are given many different names including *weak order*. When a weak order is antisymmetric, it is called a *total* or *simple order*. Cantor showed that necessary and sufficient conditions for $\langle X, \succsim \rangle$ to be represented in $\langle \text{Re}, \geqslant \rangle$ are that $\succsim$ be a weak order and there be a finite or countable dense subset Y of X that is *order dense* in X (i.e. for each $x \succ z$, there exists a y in Y such that $x \succ y \succ z$). For many purposes, this subset plays the same role as do the rational numbers within the system of real numbers.

In order for the representation to be onto either $\langle \text{Re}, \geqslant \rangle$ or $\langle \text{Re}^+, \geqslant \rangle$, which is often the case in physical measurement, two additional conditions are necessary and sufficient: *Dedekind completeness* (each non-empty bounded subset of X has a least upper bound in X) and *unboundedness* (there is neither a least nor a greatest element).

In measurement axiomatizations, one usually does not postulate a countable, order-dense subset, but derives it from axioms that are intuitively more natural. For example, when there is a binary operation of combining objects, it follows from a number of properties including an Archimedean axiom which states that no object is infinitely larger than or infinitesimally close to another object. When the structure is Dedekind complete and the operation is monotonic, it is also Archimedean. Dedekind completeness and Archimedeaness are what logicians call 'second order axioms', and in principle they are incapable of direct empirical verification.

The most fruitful and intensively examined measurement structures are those with an associative, positive binary operation. This has been the basis of most physical measurement. It has been apparent for some time that few important phenomena of the behavioural and social sciences can be modelled, directly or indirectly, in this way. The development of a general non-associative and non-positive theory began in 1976, and it is moderately well understood in certain symmetric situations. This, and its specialization to associative structures, is the focus of the rest of this entry.

3. SCALE TYPES

Classification. As was noted in the examples, scale type has to do with mappings of one numerical representation of a structure into other equally good ones. For some fixed numerical structure $\mathscr{R}$, a *scale* of the structure $\mathscr{X}$ is the collection of all $\mathscr{R}$-representations of $\mathscr{X}$. Much the simplest case, the one to which we confine most of our attention, occurs when $\mathscr{X}$ is totally ordered, the domain of $\mathscr{R}$ is either Re or Re^+, and the $\mathscr{R}$-presentations are all *onto* the domain and so are isomorphisms. Such scales are then usually described in terms of the group of real transformations that take one representation into another. As Stevens noted, four distinct groups of transformations have appeared in physical measurement: any strictly increasing function, any linear function $rx + s$, $r > 0$, any *similarity* transformation rx, $r > 0$, and the identity map. The corresponding scales are called *ordinal, interval, ratio*, and *absolute*. (Throughout this entry, ratio scales are assumed to be onto Re^+ thereby ruling out cases were an object is assigned the number 0.)

The first three scale types exhibit a property called *homogeneity*, namely, that for each element x in the qualitative structure and each real number r in the domain of $\mathscr{R}$, there is a representation that maps x into r. Homogeneity is typical of physical measurement and it plays an important role in the formulation of many physical laws. We may ask two general types of questions about it: what are the possible groups associated with homogeneous scales, and what are the general classes of structures that can yield homogeneous scales?

It is easiest to formulate answers to these questions in terms of automorphisms of the qualitative structures, that is, in terms of isomorphisms of the structure onto itself. The elements of the scale and the automorphisms of the structure are in one-to-one correspondence, since if ϕ and ψ are two representations and juxtaposition denotes function composition, then $\psi^{-1}\phi$ is an automorphism, and if ϕ is a representation and α is an automorphism, then, $\psi = \phi\alpha$ is a representation.

It is not difficult to see that homogeneity of a scale simply corresponds to there being an automorphism that takes any element of the domain of the structure into any other element. This can be made more specific. Let M be a positive integer, then $\mathscr{X}$ is said to be *M-point homogeneous* iff each strictly ordered set of M points can be mapped by an automorphism into any other strictly ordered set of M points. A structure that is not homogeneous for any positive M is said to be *0-point homogeneous*; one that is homogeneous for every finite M is said to be *∞-point homogeneous*.

Another important feature of a scale is its degree of redundancy, which we may formulate as follows: a scale is said to be *N-point unique*, where N is a non-negative integer, iff for every two representations ϕ and ψ in the scale, if ϕ and ψ agree at N distinct points, then $\psi = \phi$. By this definition, ratio scales are 1-point unique, interval scales are 2-point unique, and absolute scales 0-point unique. Scales, like ordinal ones, that take infinitely many points to determine a representation are said to be *∞-point unique*. Equally, we speak of the structure being *N-point unique* iff every two automorphisms that agree at N distinct points are identical, and it is *∞-point unique* iff it is not N-point unique for any non-negative N.

The abstract concept of scale type can be given in terms of these concepts. The *scale type* of $\mathscr{X}$ is the pair (M, N) such that M is the maximum degree of homogeneity and N is the minimum degree of uniqueness of $\mathscr{X}$. For the cases under consideration, it can be shown that $M \leqslant N$. Ratio scales are of type $(1, 1)$ and interval scales of type $(2, 2)$. Narens (1981a,b) showed that the converses of both statements are true. And Alper (1987) showed that if $M > 0$ and $N < \infty$, then $N = 1$ or 2. The group in the $(1, 2)$ case consists of transformations of the form $rx + s$, where s is any real number and r is any element of a non-trivial, proper subgroup of the group $\langle \mathrm{Re}^+, \cdot \rangle$. One example is $r = k^n$, where $k > 0$ is fixed and n ranges over the integers. So a structure is homogeneous iff it is of type $(1, 1)$, $(1, 2)$, $(2, 2)$, or (M, ∞). In the latter case, it is not known which values of M, aside from ∞, can occur. The ordinal case is (∞, ∞). We focus on the first three cases.

Unit Representation of Homogeneous Concatenation Structures. Given that we know the possible homogeneous scale types, the next question is: Which structures have scales of those types? The answer is not known completely, but for ordered structures with binary operations it is completely understood. This is useful since, as was noted, they play a central role in much physical measurement and, as we shall see below, they arise naturally in two distinct ways of interest to social scientists.

Consider real concatenation structures of the form $\mathscr{R} = \langle \mathrm{Re}^+, \geqslant, *' \rangle$, where $\geqslant$ has its usual meaning and we have replaced $+$ by a general binary, numerical operation, denoted $*'$, that is strictly increasing in each variable. The major result is that if $\mathscr{X}$ satisfies $M > 0$ and $N < \infty$ – a sufficient condition for the latter is that $*'$ be continuous (Luce and Narens, 1985) – then the structure can be mapped canonically into an isomorphic one that is of the form $\langle \mathrm{Re}^+, \geqslant, * \rangle$, where there is a function f from Re^+ onto Re^+ such that (i) f is strictly increasing, (ii) $f(x)/x$ is strictly decreasing, and (iii) for all x, y in Re^+, $x * y = yf(x/y)$ (Cohen and Narens, 1979). This type of canonical representation is called a *unit representation*. Observe that it is invariant under the similarities of a ratio scale:

$$rx * ry = ryf(rx/ry) = r[yf(x/y)] = r(x * y).$$

The two most familiar examples of unit representations are ordinary additivity, for which $f(z) = 1 + z$ and so $x * y = x + y$, and bisymmetry, for which $f(z) = z^c$ and so $x * y = x^c y^{1-c}$. Situations where such representations arise are discussed later.

The three different scale types can be distinguished by means of a simple property of the function f (Luce and Narens, 1985). Consider the values of $\rho > 0$ for which $f(x^\rho) = f(x)^\rho$ for all $x > 0$. The structure is of scale type $(1,1)$ iff $\rho = 1$; of type $(1,2)$ iff for some fixed $k > 0$ and all integers n, $\rho = k^n$; and of type $(2,2)$ iff $\rho > 0$. The $(2,2)$ condition imposes a very tight constraint on f, namely, that there be constants c, d in $(0,1)$ such that

$$f(z) = \begin{cases} z^c, & \text{if } z > 1 \\ 1, & \text{if } z = 1 \\ z^d, & \text{if } z < 1. \end{cases}$$

If, as is the usual practice in the social sciences but not in physics, we construct the structure on Re by taking logarithms, the case of the $(2, 2)$ operation becomes

$$x * y = \begin{cases} cx + (1 - c)y, & \text{if } x > y \\ x, & \text{if } x = y \\ dx + (1 - d)y, & \text{if } x < y. \end{cases}$$

Structures leading to this *dual bilinear* representation are called *dual bisymmetric* (when $c = d$, the 'dual' is dropped). They lead to an interesting generalization of the theory of subjective expected utility for gambles (section 6).

4. AXIOMATIZATION OF CONCATENATION STRUCTURES

Given this understanding of the possible representations of homogeneous, finitely unique concatenation structures, it is natural to return to the classical question of axiomatizing the qualitative properties that lead to such representations. Until a few years ago, the only two cases that were understood axiomatically were those leading to additivity and averaging (see below). We know more now, although our knowledge remains incomplete.

Additive Representations. The key mathematical result underlying extensive measurement, due to O. Hölder, states that when a group operation and an ordering interlock so that the operation is monotonic and is Archimedean in the sense that sufficient copies of any positive element will exceed any fixed element, then the group is isomorphic to an ordered subgroup of the additive real numbers. Basically, the theory of extensive measurement restricts itself to the positive subsemigroup of such a structure. Extensive structures can be shown to be of scale type (1, 1). Various generalizations involving partial operations (defined for only some pairs of objects) have been developed. (For a summary, see Krantz et al., 1971, chs 2, 3, and 5.) Not only are these structures more realistic, they are essential to an understanding of the partial additivity that arises in probability structures. These structures can be shown to be of scale type (0, 1).

The representation theory for extensive structures not only asserts the existence of a numerical representation, but provides a systematic procedure (involving the Archimedean property) for constructing one to any preassigned degree of accuracy. This construction, directly or indirectly, underlies the extensive scales used in practice.

The second classical case, due to J. Pfanzagl, leads to weighted average representations. The conditions are monotonicity of the operation, a form of solvability, an Archimedean condition, and bisymmetry $[(x \circ y) \circ (u \circ v) \sim (x \circ u) \circ (y \circ v)]$, which replaces associativity. One method of developing these representations involves two steps: first the bisymmetric operation is recoded as a conjoint one (see section 5) as follows: $(u, v) \succsim' (x, y)$ iff $u \circ v \succsim x \circ y$; and second, the conjoint structure is recoded as an extensive operation on one of its components. This reduces the proof of the representation theorem to that of extensive measurement, that is to Hölder's theorem, and so it too is constructive.

NON-ADDITIVE REPRESENTATIONS

The most completely understood generalization of extensive structures, called *positive concatenation structures* or PCSs for short, simply drops the assumption of associativity. Narens and Luce (see Narens, 1985) showed that this was sufficient to get a numerical representation and that, under a slight restriction which has since been removed, the structure is 1-point unique, but not necessarily 1-point homogeneous. Indeed, Cohen and Narens (1979) showed that the automorphism group is an Archimedean ordered group and so is isomorphic to a subgroup of the additive real numbers; it is homogeneous only when the isomorphism is to the full group. As in the extensive case, one can use the Archimedean axiom to construct representations, but the general case is a good deal more complex than the extensive one and almost certainly will require computer assistance to make it practical.

For Dedekind complete PCSs that map onto Re^+ there exists a nice criterion for 1-point homogeneity, namely, that for each positive integer and every x and y, $n(x \circ y) = nx \circ ny$, where by definition $1x = x$ and $nx = (n-1)x \circ x$. The form of the representations of all such homogeneous representations was described earlier.

The remaining broad type of concatenation structures consists of those that are idempotent, i.e. for all x, $x \circ x = x$. The following conditions have been shown to be sufficient for idempotent structures to have a numerical representation (Luce and Narens, 1985): o is an operation that is monotonic and satisfies an Archimedean condition (for differences) and a solvability condition that says for each x and y, there exist u and v such that $u \circ x = y = x \circ v$. If, in addition, such a structure is Dedekind complete, it can be shown that it is N-point unique with $N \leqslant 2$.

5. AXIOMATIZATION OF CONJOINT STRUCTURES

A second major class of measurement structures, widely familiar from both physics and the social sciences, are those, based on two or more independent variables effecting a tradeoff in the to-be-measured dependent variable. The familiar physical relations among three basic attributes, such as kinetic energy $= mv^2/2$, where m is the mass and v the velocity of a moving body, illustrates both their commonness and importance in physics. Such conjoint structures are equally common in the behavioural and social sciences: preference between commodity bundles or between gambles; loudness of pure tones as a function of signal intensity and frequency; tradeoff between delay and amount of a reward etc. Although there is some theory for more than two independent variables in the additive case, for present purposes we confine attention to the two variable case $\langle X \times Y, \succsim \rangle$.

As for concatenation structures, the simplest case to understand is the additive one in which the major non-structural properties are:

(i) Independence (monotonicity): if $(x, y) \succsim (x', y)$ holds for some y then it holds for all y in Y, and the parallel statement for the other component. Note that this property allows us to induce natural orderings, $\succsim_X$ and $\succsim_Y$, on X and Y.

(ii) Thomsen condition: if $(x, z) \sim (u, y')$ and $(u, y) \sim (x', z)$, then $(x, y) \sim (x', y')$.

(iii) An Archimedean condition which says, for each component, if $\{x_i\}$ is a bounded sequence and for some non-equivalent y and z it satisfies $(x_i, y) \sim (x_{i+1}, z)$, then the sequence is finite.

These, together with some solvability in the structure, are sufficient to prove the existence of an interval scale, additive representation (for a summary of various results, see Krantz et al., 1971, chs 6, 7 and 9). The result has been generalized to non-additive representations by dropping the Thomsen condition, which leads to the existence of a non-additive numerical representation. The basic strategy is to define on one component, say X, an operation $*_X$ that captures the information embodied in the tradeoff between components. The induced structure $\langle X, \succsim_X, *_X \rangle$ can be shown to consist of two PCSs pieced together at an element that acts like a natural zero of the concatenation structure. The results for PCSs are then used to construct the representation. As might be anticipated, $*_X$ is associative if and only if the conjoint structure satisfies the Thomsen condition.

The important case of a conjoint structure having an operation on one of its components that is coupled to the conjoint structure by means of a distribution law is taken up in section 4 of MEANINGFULNESS AND INVARIANCE.

6. GAMBLING STRUCTURES

Rationality Assumptions in the Traditional Theory. As was noted earlier, an extensive literature on preferences among gambles exists. The major theoretical development is the axiomatization of subjective expected utility (SEU), which is a representation satisfying equation (1). Although such axiomatizations are defensible theories in terms of principles of rationality, they fail as descriptions of human behaviour. The rationality axioms invoked are of three quite distinct types.

First, preference is assumed to be transitive. This assumption has been shown to fail in various empirical contexts (especially multifactor ones), with perhaps the most pervasive and still ill-understood example being the 'preference reversal phenomenon', discovered by P. Slovic and S. Lichtenstein and investigated extensively by, among others, Grether and Plott

(1979) (see references there to the earlier work). Nevertheless, transitivity is the axiom that is least easily given up. Even subjects who violate it are not inclined to defend their 'errors'. A few attempts have been made to develop theories without it, but so far they are complex and have not received much empirical scrutiny (Bell, 1982; Fishburn, 1982, 1985).

The second type of rationality postulates 'accounting' principles in which two gambles are asserted to be equivalent in preference because when analysed into their component outcomes they are seen to be identical. For example, if $x \circ_A y$ is a gamble and $(x \circ_A y) \circ_B z$ means that the event B occurs first and then, independent of it, A occurs, then on accounting grounds $(x \circ_A y) \circ_B y \sim (x \circ_B y) \circ_A y$ is rational, since on both sides x is the outcome when A and B both occur (though in opposite orders) and y otherwise. One of the first 'paradoxes' of utility theory, that of M. Allais, is a violation of an accounting equation which assumes that certain probability calculations also take place.

The third type of rationality condition is the extended sure-thing principle, equation (2). Its failure, which occurs regularly in experiments, is substantially the 'paradox' earlier pointed out by D. Ellsberg. Subjects have insisted on the reasonableness of their violations of this principle (MacCrimmon, 1967).

Some Generalizations of SEU. Kahneman and Tversky (1979) proposed a modification of the expected utility representation designed to accommodate the last two types of violations. Luce and Narens (1985) developed a somewhat related and more comprehensive theory, based on the dual bilinear representation described above. The representation takes the form:

$$U(xo_A y) = \begin{cases} S^+(A)U(x) + [1 - S^+(A)]U(y), & \\ & \text{if } U(x) > U(y) \\ U(x), & \text{if } U(x) = U(y) \\ S^-(A)U(x) + [1 - S^-(A)]U(y), & \\ & \text{if } U(x) < U(y), \end{cases}$$

where the S^i are weights, not necessarily probabilities. In such a structure, the accounting equation $(x \circ_A y) \circ_B y \sim (x \circ_B y) \circ_A y$ mentioned above necessarily holds. Another simple and often postulated accounting equation is $x \circ_A y \sim y \circ_{\sim A} x$, which holds in the model iff $S^+(A) + S^-(\sim A) = 1$. They show that any further accounting equations not derived from the latter equation and the model force the bisymmetric case, i.e., $S^+ = S^-$. The extended sure-thing principle, which is not an accounting equation, is equivalent to: for events A, B, C with C disjoint from A and B and $i = +, -$,

$$S^i(A) \geqslant S^i(B) \quad \text{iff} \quad S^i(A \cup C) \geqslant S^i(B \cup C),$$

which of course is true when the S's are probability measures. It follows easily that if the accounting equation $x \circ_A y \sim y \circ_{\sim A} x$ holds and if the $S^i, i = +, -$, are probability functions, then $S^+ = S^-$, and so the SEU model holds.

No axiomatic justification has yet been given for this model, and it has yet to be subjected to searching empirical criticism. However, it does predict most of the empirical failures of the SEU model.

Another interesting line of development, involving a different weighting than in traditional SEU, can be found in Chew (1980, 1983).

R. Duncan Luce and Louis Narens

See also DIMENSIONS OF ECONOMIC QUANTITIES; MEANINGFULNESS AND INVARIANCE; MEAN VALUE; TRANSFORMATIONS AND INVARIANCE.

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